Quantum Field Theory

Exercises in preparation for the exam 3: solutions

Exercise 1: two real scalar fields

a) **This was already solved in Homework 2** In order to find the physical spectrum, one has to diagonalize the kinetic term. In order to do so, define:

$$\begin{cases} \phi_1 = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \\ \phi_2 = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2) \end{cases} \quad \text{or equivalently} \quad \begin{cases} \varphi_1 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) \\ \varphi_2 = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2) \end{cases} \tag{1}$$

The different terms in the Lagrangian become

$$\begin{aligned} \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 \to & \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 \\ g\partial_\mu\varphi_1\partial^\mu\varphi_2 \to & \frac{g}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 - \partial_\mu\phi_2\partial^\mu\phi_2) \\ (\varphi_1^2 + \varphi_2^2) \to & (\phi_1^2 + \phi_2^2) \end{aligned}$$

So :

$$\mathcal{L} = \frac{1}{2}(1+g)\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}(1-g)\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{m^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2 \tag{2}$$

In order to have a physically acceptable theory, we need both kinetic terms to be positive (see homework 1), so :

$$|g| < 1 \tag{3}$$

In order to find the physical masses, we canonically normalize the fields:

$$\Phi_1 = \frac{\phi_1}{\sqrt{1+g}} \quad \text{and} \quad \Phi_2 = \frac{\phi_2}{\sqrt{1-g}} \tag{4}$$

and we find :

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi_1\partial^\mu\Phi_1 + \frac{1}{2}\partial_\mu\Phi_2\partial^\mu\Phi_2 + \frac{m^2\Phi_1^2}{2(1+g)} + \frac{m^2\Phi_2^2}{2(1-g)} + \frac{\lambda\Phi_1^4}{4!(1-g)^2} + \frac{\lambda\Phi_2^4}{4!(1-g)^2} + \frac{2\lambda\Phi_1^2\Phi_2^2}{4!(1-g^2)} \tag{5}$$

So we have two physical particles of masses $m_1^2 = \frac{m^2}{1+g}$ and $m_2^2 = \frac{m^2}{1-g}$.

Note : The case $g = \pm 1$ is more subtle because the theory becomes strogly coupled and will not be treated here. However, when $\lambda = 0$, then we are left with (in the $g = 1$ case) :

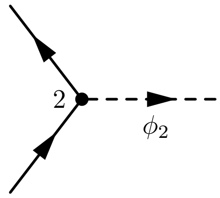
$$\mathcal{L} = \partial_\mu\phi_1\partial^\mu\phi_1 + \frac{m^2}{2}(\phi_1^2 + \phi_2^2) \tag{6}$$

ϕ_2 has no kinetic term and is called an auxiliary field. We can solve explicitly the equations of motion and get : $\phi_2 = 0$. We are left with one physical particle of mass $\frac{m}{\sqrt{2}}$.

b) We will consider the case $|g| < 1$ and compute the cross-section for the process $\Phi_2(p_1)\Phi_2(p_2) \to \Phi_1(q_1)\Phi_1(q_2)$. The interaction Lagrangian relevant at tree-level for this process is :

$$\mathcal{L}_{int} = \frac{\lambda\Phi_1^2\Phi_2^2}{12(1-g^2)} \tag{7}$$

There is only one Feynman diagram to consider :



$$= i\frac{\lambda}{12(1-g^2)} \times 4 = i\mathcal{M}. \tag{8}$$

The square is immediate to take :

$$|\mathcal{M}|^2 = \frac{\lambda^2}{9(1-g^2)^2}\,, \tag{9}$$

and inserting this expression in the formula for the cross-section of a two to two scattering process in the center of mass frame:

$$d\sigma = \frac{1}{2\sqrt{(s-2m_2^2)^2 - 4m_2^4}}|\mathcal{M}|^2\sqrt{1-\frac{4m_1^2}{s}}\frac{d\Omega}{32\pi^2} \tag{10}$$

we get the cross section by integrating over the angular variable (don't forget the factor 1/2 because of identical particles in the final state) :

$$\sigma(\Phi_2\Phi_2 \to \Phi_1\Phi_1) = \frac{1}{288\pi}\frac{\lambda^2}{(1-g^2)^2}\frac{1}{\sqrt{s\left(s-4\frac{m^2}{1-g}\right)}}\sqrt{1-\frac{4m^2}{s(1+g)}}\,. \tag{11}$$

Exercise 2: N fermions and 1 scalar with $U(N)$ symmetry

The building block for constructing the required Lagrangian are

$$\left\{\sum_{a=1}^{n}\bar{\psi}_a(x)(\Gamma/\partial)\psi_a(x),\ \phi(x)\right\} \tag{12}$$

and derivatives of these. Here $\Gamma = \{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}\}$ and ∂ denotes any possible derivative. Furthermore, we known that the canonical dimensions of the fields are:

$$[\phi] = 1, \qquad [\psi_a] = 3/2. \tag{13}$$

Then, recalling that derivatives have mass dimension 1, it is easy to realize that the most general Lagrangian with canonical kinetic terms, whose operators have canonical dimension $d \leq 4$ is, up to irrelevant constants,

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^2}{2}\phi^2 + \sum_{a=1}^{n}\bar{\psi}_a(i\partial\!\!\!/ - m_\psi)\psi_a + i\,\mu\sum_{a=1}^{n}\bar{\psi}_a\gamma_5\psi_a + c\phi \\ &+ a_1\phi^3 + a_2\phi^4 + b_1\phi\sum_{a=1}^{n}\bar{\psi}_a\psi_a + i\,b_2\phi\sum_{a=1}^{n}\bar{\psi}_a\gamma_5\psi_a.\end{aligned} \tag{14}$$

The i factors ensure that all parameters are real. We can always shift the definition of the scalar field as

$$\phi \to \phi + \frac{c}{m_\phi^2}, \tag{15}$$

so that the linear term $c\phi$ disappear. We can also get rid of the term $\sum_{a=1}^{n}\bar{\psi}_a\gamma_5\psi_a$. It is easy to see this using the Weyl decomposition of a Dirac field:

$$\psi = \begin{pmatrix}\eta_\alpha \\ \bar{\chi}^{\dot{\alpha}}\end{pmatrix}, \tag{16}$$

which gives:

$$\bar{\psi}(m_\psi - i\mu\gamma_5)\psi = (m_\psi - i\mu)\chi\eta + (m_\psi + i\mu)\bar{\chi}\bar{\eta} = Me^{-i\alpha}\chi\eta + Me^{+i\alpha}\bar{\chi}\bar{\eta},$$
$$M = m_\psi^2 + \mu^2, \qquad \alpha = \arctan(\mu/M). \tag{17}$$

Then via a chiral rotation we can rewrite it in canonical form

$$\eta \longrightarrow e^{i\alpha/2}\eta, \quad \chi \longrightarrow e^{i\alpha/2}\chi \quad \Longrightarrow \quad \bar{\psi}(m_\psi - i\mu\gamma_5)\psi \longrightarrow M\chi\eta + M\bar{\chi}\bar{\eta} = \bar{\psi}M\psi. \tag{18}$$

This can also be seen working solely in Dirac notation. Indeed a chiral rotation acts on a Dirac field as[1]

$$\psi \to e^{i\alpha\gamma_5}\psi = \cos\alpha\,\psi + i\sin\alpha\,\gamma_5\,\psi. \tag{19}$$

Such a transformation leaves the kinetic term unchanged. The mass terms are modified as follows:

$$\begin{aligned}\bar{\psi}(m_\psi - i\mu\gamma_5)\psi \to &(\cos\alpha\bar{\psi} + i\sin\alpha\bar{\psi}\gamma_5)(m_\psi - i\mu\gamma_5)(\cos\alpha\psi + i\sin\alpha\gamma_5\psi)\\ &= \left[(\cos^2\alpha - \sin^2\alpha)m_\psi + 2\cos\alpha\sin\alpha\,\mu\right]\bar{\psi}\psi\\ &\quad + i\left[2\sin\alpha\cos\alpha\,m_\psi - \mu(\cos^2\alpha - \sin^2\alpha)\right]\bar{\psi}\gamma_5\psi.\end{aligned} \tag{20}$$

Then, choosing $\alpha = \arctan(\mu/m)/2$, we erase the second term. Notice that, as long as $b_1/b_2 \neq \mu/m$ we do not modify the structure of the interaction Lagrangian. Then, we proved that the most general Lagrangian we can write is[2]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_\phi^2}{2}\phi^2 + \sum_{a=1}^{n}\bar{\psi}_a(i\partial\!\!\!/ - m_\psi)\psi_a + a_1\phi^3 + a_2\phi^4 + b_1\phi\sum_{a=1}^{n}\bar{\psi}_a\psi_a + i\,b_2\phi\sum_{a=1}^{n}\bar{\psi}_a\gamma_5\psi_a. \tag{21}$$

If no term vanishes, there is no parity assignment for ϕ which makes this Lagrangian parity invariant. Indeed supposing

$$\phi \xrightarrow{P} \eta\,\phi, \tag{22}$$

the quadratic Lagrangian is unchanged, but we have:

$$\phi^3 \xrightarrow{P} \eta\,\phi^3, \qquad \phi^4 \xrightarrow{P} \phi^4, \tag{23}$$

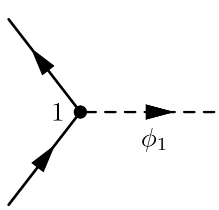
$$\phi\sum_{a=1}^{n}\bar{\psi}_a\psi_a \xrightarrow{P} \eta\,\phi\sum_{a=1}^{n}\bar{\psi}_a\psi_a, \qquad \phi\sum_{a=1}^{n}\bar{\psi}_a\gamma_5\psi_a \xrightarrow{P} -\eta\,\phi\sum_{a=1}^{n}\bar{\psi}_a\gamma_5\psi_a. \tag{24}$$

This also shows that the Lagrangian is parity invariant choosing $\eta = 1$ if $b_2 = 0$, but it can also be parity invariant with $\eta = -1$ if $b_1 = a_1 = 0$.

Now suppose $m_\phi > 2m_\psi$. We can compute the decay rate of ϕ. The relevant interaction term is given by:

$$\mathcal{L}_{int} \supset \phi\sum_{a=1}^{n}\bar{\psi}_a(b_1 + i\,b_2\gamma_5)\psi_a. \tag{25}$$

The leading order contribution to the decay comes from the N processes $\phi \to \psi_a\bar{\psi}_a$. There is then one Feyman diagram which contributes to the matrix element at leading order:



$$= i\bar{u}^r(k)(b_1 + i\,b_2\gamma_5)v^s(q) = i\mathcal{M}, \tag{26}$$

[1] To prove this formula, expand $e^{i\alpha\gamma_5} = 1 + i\alpha\gamma_5 - \frac{1}{2}\gamma_5^2 \ldots$ and use $\gamma_5^2 = 1$; finally compare with the Taylor expansion of sin and cos.

[2] The coefficients appearing in (21) are in general different than thos in (14).

where r, s label the spin of the final particles. Then we get:

$$|\mathcal{M}|^2 = [\bar{v}^s(q)(b_1 + i\,b_2\gamma_5)u^r(k)][\bar{u}^r(k)(b_1 + i\,b_2\gamma_5)v^s(q)]. \tag{27}$$

To compute the total decay rate, we sum over all final possible polarizations and we multipy by N to account of all possible final states $\psi_a\,\bar{\psi}_a$, $a = 1, \ldots, N$:

$$\overline{|\mathcal{M}|}^2 = N\sum_{r,s}|\mathcal{M}|^2 = N\mathrm{Tr}\left[(\not{q} - m_\psi)(b_1 + i\,b_2\gamma_5)(\not{k} + m_\psi)(b_1 + i\,b_2\gamma_5)\right], \tag{28}$$

where we used $\sum_r u^r(k)\bar{u}^r(k) = (\not{k} + m_\psi)$ and $\sum_s v^s(q)\bar{v}^s(q) = (\not{q} - m_\psi)$. The trace is evaluated in the standard way:

$$\overline{|\mathcal{M}|}^2 = 4N\left[(b_1^2 + b_2^2)q\cdot k - (b_1^2 - b_2^2)m_\psi^2\right]. \tag{29}$$

Using then

$$m_\phi^2 = p^2 = (q+k)^2 = 2m_\psi^2 + 2q\cdot k \qquad \Longrightarrow \qquad q\cdot k = \frac{m_\phi^2}{2} - m_\psi^2, \tag{30}$$

we find

$$\overline{|\mathcal{M}|}^2 = 4N\left[b_1^2\left(\frac{m_\phi^2}{2} - 2m_\psi^2\right) + b_2^2\frac{m_\phi^2}{2}\right]. \tag{31}$$

The decay rate then follows. In the CM frame:

$$\Gamma = \frac{\overline{|\mathcal{M}|}^2}{16\pi\,m_\phi}\sqrt{1 - \frac{4m_\psi^2}{m_\phi^2}} = \frac{4N\left[b_1^2\left(\frac{m_\phi^2}{2} - 2m_\psi^2\right) + b_2^2\frac{m_\phi^2}{2}\right]}{16\pi\,m_\phi}\sqrt{1 - \frac{4m_\psi^2}{m_\phi^2}}. \tag{32}$$

In the limit $N \to \infty$ this diverges, as it is to be expected since ϕ can decay into N different pairs of particles. What happens is that in the $N \to \infty$ limit the real coupling constants of the system are $\sqrt{N}b_1$ and $\sqrt{N}b_2$, as the previous computation suggests. The limit $N \to \infty$ with g fixed hence corresponds to a regime where the theory becomes strongly coupled and the perturbative expansion breaks. The decay rate is instead well defined in the limit $N \to \infty$, $b_1 \to 0, b_2 \to 0$ with $Nb_1^2 =$ fixed and $Nb_2^2 =$ fixed.

Exercise 3: modified $O(2)$ model

In the case $g = 0$ the Lagrangian is manifestly invariant under $O(2)$ rotations:

$$\phi_i \to \phi_i' = O_{ij}\phi_j, \qquad O \in O(2). \tag{33}$$

To analyze the case $g \neq 0$ it is conveneint to use the Weyl decomposition of a Dirac field:

$$\psi = \begin{pmatrix} \eta_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{34}$$

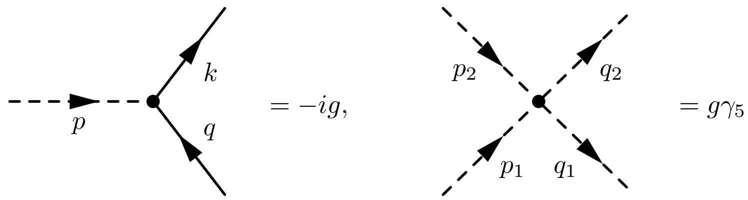
which gives:

$$\bar{\psi}(\phi_1 + i\gamma_5\phi_2)\psi = (\phi_1 + i\phi_2)\chi\eta + (\phi_1 - i\phi_2)\bar{\chi}\bar{\eta} = \sqrt{\left(\sum_{i=1}^{2}\phi_i^2\right)}\left(e^{i\alpha}\chi\eta + e^{-i\alpha}\bar{\chi}\bar{\eta}\right), \tag{35}$$

where $\alpha = \arctan(\phi_1/\phi_2)$. Then the Lagrangian is invariant under a combined $O(2)$ rotation of the scalar fields and a chiral rotation of the Dirac field

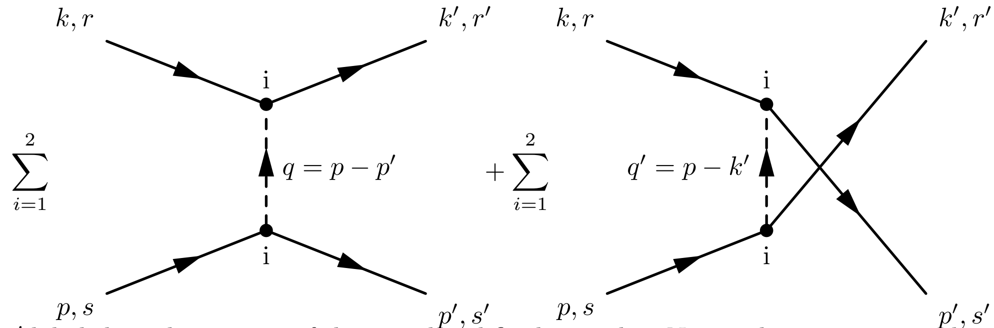
$$\phi_i \to \phi_i' = O_{ij}\phi_j, \qquad \psi \to \psi' = e^{-i\gamma_5(\alpha'-\alpha)/2}\psi, \quad \alpha' = \arctan(\phi_1'/\phi_2'). \tag{36}$$

There are two relevant interaction vertices:



$$(37)$$

For each of these vertices, both t- and u-channel contribute to the process $\psi\psi \to \psi\psi$. The the tree-level amplitude receives contributions from 4 Feynman diagrams (time flows from left to right):



here s, s', r, r' label the polarizations of the initial and final particles. Notice the sum over i. Then the matrix element reads:

$$\begin{aligned} i\mathcal{M} = &- g^2 \frac{i}{(p-p')^2 - m^2} \left[\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\right] \\ &+ g^2 \frac{i}{(p-p')^2 - m^2} \left[\bar{u}(p',s')\gamma_5 u(p,s)\right]\left[\bar{u}(k',r')\gamma_5 u(k,r)\right] \\ &+ g^2 \frac{i}{(p-k')^2 - m^2} \left[\bar{u}(k',r')u(p,s)\right]\left[\bar{u}(p',s')u(k,r)\right] \\ &- g^2 \frac{i}{(p-k')^2 - m^2} \left[\bar{u}(k',r')\gamma_5 u(p,s)\right]\left[\bar{u}(p',s')\gamma_5 u(k,r)\right] \end{aligned} \tag{38}$$

Notice that there is a minus sign difference between t and u channel.

We now need to square this matrix element, average over initial polarizations and sum over the final ones:

$$|\overline{\mathcal{M}}|^2 \equiv \frac{1}{4}\sum_{s,r}\sum_{s',r'} |\mathcal{M}|^2. \tag{39}$$

In doing this operation we have $4 \times 4 = 16$ terms. Four terms are just the square of each diagram:

$$\begin{aligned} &\sum_{s,r}\sum_{s',r'} \left[\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\right]\left\{\left[\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\right]\right\}^\dagger \\ &= \sum_{s,r}\sum_{s',r'} \left[\bar{u}(p,s)u(p',s')\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\bar{u}(k,r)u(k',r')\right] \\ &= \mathrm{Tr}\left[\not{p}\not{p}'\right]\mathrm{Tr}\left[\not{k}\not{k}'\right] = 16(p\cdot p')(k\cdot k'), \end{aligned} \tag{40}$$

$$\begin{aligned} &\sum_{s,r}\sum_{s',r'} \left[\bar{u}(p',s')\gamma_5 u(p,s)\right]\left[\bar{u}(k',r')\gamma_5 u(k,r)\right]\left\{\left[\bar{u}(p',s')\gamma_5 u(p,s)\right]\left[\bar{u}(k',r')\gamma_5 u(k,r)\right]\right\}^\dagger \\ &= \sum_{s,r}\sum_{s',r'} \left[\bar{u}(p,s)\gamma_5 u(p',s')\bar{u}(p',s')\gamma_5 u(p,s)\right]\left[\bar{u}(k',r')\gamma_5 u(k,r)\bar{u}(k,r)\gamma_5 u(k',r')\right] \\ &= \mathrm{Tr}\left[\not{p}\gamma_5\not{p}'\gamma_5\right]\mathrm{Tr}\left[\not{k}\gamma_5\not{k}'\gamma_5\right] = 16(p\cdot p')(k\cdot k'), \end{aligned} \tag{41}$$

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(k',r')u(p,s)\right]\left[\bar{u}(p',s')u(k,r)\right]\left\{\left[\bar{u}(k',r')u(p,s)\right]\left[\bar{u}(p',s')u(k,r)\right]\right\}^\dagger \\
&= \ldots = \mathrm{Tr}\left[\not{p}\not{k}'\right]\mathrm{Tr}\left[\not{k}\not{p}'\right] = 16(p\cdot k')(k\cdot p'),
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(k',r')\gamma_5 u(p,s)\right]\left[\bar{u}(p',s')\gamma_5 u(k,r)\right]\left\{\left[\bar{u}(k',r')\gamma_5 u(p,s)\right]\left[\bar{u}(p',s')\gamma_5 u(k,r)\right]\right\}^\dagger \\
&= \ldots = \mathrm{Tr}\left[\not{p}\gamma_5\not{k}'\gamma_5\right]\mathrm{Tr}\left[\gamma_5\not{k}\gamma_5\not{p}'\right] = 16(p\cdot k')(k\cdot p').
\end{aligned} \tag{43}
$$

The remaining terms are just twice the six possible cross terms. Two of them vanish:

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\right]\left\{\left[\bar{u}(p',s')\gamma_5 u(p,s)\right]\left[\bar{u}(k',r')\gamma_5 u(k,r)\right]\right\}^\dagger \\
&= \sum_{s,r}\sum_{s',r'}\left[\bar{u}(p,s)\gamma_5 u(p',s')\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\bar{u}(k,r)\gamma_5 u(k',r')\right] \\
&= \mathrm{Tr}\left[\not{p}\gamma_5\not{p}'\right]\mathrm{Tr}\left[\not{k}\gamma_5\not{k}'\right] = 0,
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(k',r')u(p,s)\right]\left[\bar{u}(p',s')u(k,r)\right]\left\{\left[\bar{u}(k',r')\gamma_5 u(p,s)\right]\left[\bar{u}(p',s')\gamma_5 u(k,r)\right]\right\}^\dagger \\
&= \ldots = \mathrm{Tr}\left[\not{p}\gamma_5\not{k}'\right]\mathrm{Tr}\left[\not{k}\gamma_5\not{p}'\right] = 0.
\end{aligned} \tag{45}
$$

The remaing four are all proportional to the same trace:

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\right]\left\{\left[\bar{u}(k',r')u(p,s)\right]\left[\bar{u}(p',s')u(k,r)\right]\right\}^\dagger \\
&= \sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')u(p,s)\bar{u}(p,s)u(k',r')\bar{u}(k',r')u(k,r)\bar{u}(k,r)u(p',s')\right] \\
&= \mathrm{Tr}\left[\not{p}'\not{p}\not{k}'\not{k}\right],
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')u(p,s)\right]\left[\bar{u}(k',r')u(k,r)\right]\left\{\left[\bar{u}(k',r')\gamma_5 u(p,s)\right]\left[\bar{u}(p',s')\gamma_5 u(k,r)\right]\right\}^\dagger \\
&= \sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')u(p,s)\bar{u}(p,s)\gamma_5 u(k',r')\bar{u}(k',r')u(k,r)\bar{u}(k,r)\gamma_5 u(p',s')\right] \\
&= \mathrm{Tr}\left[\not{p}'\not{p}\gamma_5\not{k}'\not{k}\gamma_5\right] = \mathrm{Tr}\left[\not{p}'\not{p}\not{k}'\not{k}\right],
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')\gamma_5 u(p,s)\right]\left[\bar{u}(k',r')\gamma_5 u(k,r)\right]\left\{\left[\bar{u}(k',r')u(p,s)\right]\left[\bar{u}(p',s')u(k,r)\right]\right\}^\dagger \\
&= \sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')\gamma_5 u(p,s)\bar{u}(p,s)u(k',r')\bar{u}(k',r')\gamma_5 u(k,r)\bar{u}(k,r)u(p',s')\right] \\
&= \mathrm{Tr}\left[\not{p}'\gamma_5\not{p}\not{k}'\gamma_5\not{k}\right] = \mathrm{Tr}\left[\not{p}'\not{p}\not{k}'\not{k}\right],
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
&\sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')\gamma_5 u(p,s)\right]\left[\bar{u}(k',r')\gamma_5 u(k,r)\right]\left\{\left[\bar{u}(k',r')\gamma_5 u(p,s)\right]\left[\bar{u}(p',s')\gamma_5 u(k,r)\right]\right\}^\dagger \\
&= \sum_{s,r}\sum_{s',r'}\left[\bar{u}(p',s')\gamma_5 u(p,s)\bar{u}(p,s)\gamma_5 u(k',r')\bar{u}(k',r')\gamma_5 u(k,r)\bar{u}(k,r)\gamma_5 u(p',s')\right] \\
&= \mathrm{Tr}\left[\not{p}'\gamma_5\not{p}\gamma_5\not{k}'\gamma_5\not{k}\gamma_5\right] = \mathrm{Tr}\left[\not{p}'\not{p}\not{k}'\not{k}\right].
\end{aligned} \tag{49}
$$

These cross products appear with alternate sign when expanding the product and hence cancel out. Finally we are left with:

$$|\overline{\mathcal{M}}|^2 = 8g^2 \left[\frac{(p \cdot p')(k \cdot k')}{\left[(p-p')^2 - m^2\right]^2} + \frac{(p \cdot k')(k \cdot p')}{\left[(p-k')^2 - m^2\right]^2} \right]. \tag{50}$$

In terms of the Mandelstam variables:

$$s = (p+k)^2 = (p'+k')^2 = 2p \cdot k = 2p' \cdot k', \tag{51}$$

$$t = (p-p')^2 = (k-k')^2 = -2p \cdot p' = -2k \cdot k', \tag{52}$$

$$u = (p-k')^2 = (k-p')^2 = -2p \cdot k' = -2k \cdot p', \tag{53}$$

we get:

$$|\overline{\mathcal{M}}|^2 = 2g^4 \left[\frac{t^2}{\left[t - m^2\right]^2} + \frac{u^2}{\left[u - m^2\right]^2} \right]. \tag{54}$$

The differential cross section in the center of mass frame (see Peskin 4.85 or solution 11) then is:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = |\overline{\mathcal{M}}|^2 \frac{d\cos\theta d\phi}{128\pi^2 s}. \tag{55}$$

Finally, for $m = 0$, we get:

$$\sigma_{tot} = \frac{g^4}{16\pi s}. \tag{56}$$

Notice that the behaviour $\sigma \sim g^4/s$ could have been guessed by dimensional analysis.